Smallest induced drag over the wing

1. Problem-

For a wing given lift, the lowest induced drag is obtained at the prescribed range if the lift is distributed along a semi-ellipse. The secondary condition that the span is prescribed is, however, absolutely essential, and the assertion that the elliptical lift distribution is the best is absolutely inadmissible. The greater the span, the smaller the induced drag. If, in a special case, the range of the aircraft is limited by the requirement that the aircraft can be pushed through a certain predetermined hall door, it is appropriate to elliptically distribute the buoyancy within the prescribed range. But if there is no such limitation, then one will have to look to other points of view. Any increase in the span is forbidden due to the weight of the spar, which in this case is too heavy. A formulation of the weight of the non-load-bearing parts, which does justice to the aeronautical issues, is to be regarded as given, and that the wing is now looking for the shape of the wing through which the total wing resistance (induced plus profile resistance), in which the spar weight also has an effect, becomes a minimum. It would be very difficult to formulate this problem as a problem of variation.

A simpler task is to be set here, but it is to be carried out precisely. One comes to a reasonable limitation of the span also by prescribing the moment of inertia Ar^2 of the lift distribution in addition to the total lift A of the wing. r is then the "radius of inertia/gyration" of the lift distribution. You are guided to the moment of inertia of the lift distribution if you set the weight of the spar at every point proportional to the bending moment M acting there. However, this would only apply exactly if the spar had the same support height everywhere and the web weight were negligible compared to the weight of the flanges. After all, it already means an approximation of what one actually wants to print out when one stipulates that a certain amount should not be exceeded with the hydrofoil

$$\int_{-b/2}^{+b/2} M\,dx$$

The double of this integral can now be surrounded by two partial integration directly into the above-mentioned moment of inertia of the lift distribution. If one expresses the buoyancy/(maybe lift?) with the help of the KUTTA-JOUKOWSKIAN theorem by the formula

$$A = \rho v \int \Gamma dx$$

then for $x > 0$ it's

$$M(x) = \rho v \int_{x}^{b/2} \Gamma(x' - x) dx'$$

and it is now

$$\int_{0}^{b/2} \mathrm{M} dx = [xM]_0^{+b/2} - \int_{0}^{b/2} x \frac{dM}{dx} dx'$$

where

$$\frac{dM}{dx} = -\rho v \int_{x}^{b/2} \Gamma dx$$

The first expression disappears here. The second partial integration now yields

$$\int_{0}^{b/2} \mathrm{M} dx = \rho v \int_{0}^{b/2} \frac{x^2}{2} \Gamma dx$$

in a simple way, which corresponds to our claim above.

Our mathematical problem should therefore be: It is to make

$$W_i = \rho \int \Gamma w dx$$

to a minimum under the secondary conditions

$$A = \rho v \int \Gamma dx = given$$

and

$$Ar^2 = \rho v \int \Gamma x^2 dx = given$$

Here, w is the downwash velocity on the wing belonging to the circulation distribution Γ according to the wing theory. Obviously, the rule that the support radius r should have a certain size also makes a statement about the transverse extent of the wing, but one which records the mean extent of the wing and does not contain any binding for the design of the wing tip. In fact, this task also results in very different lift distributions than the elliptical one.

2. Implementation-

The task is now divided into two parts. Firstly, the problem of variation has to be solved, which tells us according to which law within the selected tension world of the lifts ?? - dividing is, and secondly a minimum problem which, in the event that the span is left completely free, selects from the solutions admissible according to the variation problem that which gives the smallest resistance at all.

The problem of variation can be solved relatively easily if we make use of an idea from A. BETZ, namely that it is permissible to apply the variations $\delta\Gamma$ to an auxiliary wing that has been shifted far backwards instead of on the wing itself and whose forward reaction is neglected can, and which is even below the downward speed $2w$. If the circulation $\Gamma(x)$ is the solution sought, then a variation of W_i must disappear by the addition of such additional circulations $\delta\Gamma$ which at the same time satisfy the secondary conditions. These constraints are apparently

$$\delta A = \rho v \int \delta\Gamma dx = 0 \quad (1)$$

and

$$\delta(Ar^2) = \rho v \int \delta\Gamma\, x^2 dx = 0. \quad (2)$$

Obviously it becomes

$$\delta W_i = \rho \int \delta \Gamma w dx = 0, \quad (3)$$

if

$$w = C_1 + C_2 x^2, \quad (4)$$

is, because then 3 is also fulfilled by the constraints 1 and 2.

The circulation associated with the value of w given in Eq. 4 is known from old developments in wing theory. If you put

$$\Gamma = \left(\Gamma_0 + \Gamma_2 \xi^2\right)\sqrt{1-\xi^2},$$

where ξ is an abbreviation for $x : \frac{b}{2}$, you get a formula for the downward velocity which is exactly of the type of Eq. 4, namely

$$C_1 = \frac{1}{2b}\left(\Gamma_0 - \frac{1}{2}\Gamma_2\right)$$

and

$$C_2 = \frac{6\Gamma_2}{b^3}.$$

This solves the problem of variation, and the next step is to find the best values of Γ_0 and Γ_2 that are compatible with the conditions of the task. Considering that Γ_2 will turn out to be negative, we introduce

$$-\Gamma_2 / \Gamma_0 = \mu$$

and thus get

$$\Gamma = \Gamma_0\left(1 - \mu \xi^2\right)\sqrt{1-\xi^2} \quad (5)$$

and

$$w = \frac{\Gamma_0}{2b}\left(1 + \frac{\mu}{2} - 3\mu \xi^2\right) \quad (6)$$

Our constraints are now

$$A = \frac{\pi}{4}\rho b v \Gamma_0\left(1 - \frac{\mu}{4}\right) \quad (7)$$

and

$$A r^2 = \frac{\pi}{64}\rho b^3 v \Gamma_0\left(1 - \frac{\mu}{2}\right) \quad (8)$$

Division immediately results in

$$r^2 = \frac{1}{16} b^2 \frac{1 - \frac{\mu}{2}}{1 - \frac{\mu}{4}}$$

or

$$b = 4r\sqrt{\frac{1 - \frac{\mu}{4}}{1 - \frac{\mu}{2}}} \quad (9)$$

By inserting i tinto Eq. 7, this gives

$$\Gamma_0 = \frac{A}{\pi\rho v r}\sqrt{\frac{1-\frac{\mu}{2}}{\left(1-\frac{\mu}{4}\right)^3}} \quad (10)$$

The induced resistance results according to S. 32 of the "Four Treatises" as

$$W_i = \frac{\pi\rho\Gamma_0^2}{8}\left(1-\frac{\mu}{2}+\frac{\mu^2}{4}\right)$$

Taking into account Eq. 10

$$W_i = \frac{A^2}{8\pi\rho v^2 r^2}\frac{\left(1-\frac{\mu}{2}\right)\left(1-\frac{\mu}{2}+\frac{\mu^2}{4}\right)}{\left(1-\frac{\mu}{4}\right)^3} \quad (11)$$

The course of the function of μ occurring here, which is to be called here for the abbreviation $f(\mu)$, can be seen from the small number table below, in which the course of $b/4r$ and that of the factor of Γ_0 are also given. It is easy to see that the minimum of $f(\mu)$ is at $\mu = 1$, which could also be demonstrated by subtracting the differential quotient of $f(\mu)$, which exactly vanishes at $\mu = 1$. However, the function here does not have an ordinary minimum, but a turning point and decreases further for the values $\mu > 1$.

Zahlentafel 1

μ	$\sqrt{\frac{1-\frac{\mu}{4}}{1-\frac{\mu}{2}}}$	$\sqrt{\frac{1-\frac{\mu}{2}}{\left(1-\frac{\mu}{4}\right)^3}}$	$\frac{\left(1-\frac{\mu}{2}\right)\left(1-\frac{\mu}{2}+\frac{\mu}{4}\right)^2}{\left(1-\frac{\mu}{4}\right)^3}$
0,00	1,0000	1,0000	1,0000
0,25	1,0351	1,0305	0,9458
0,50	1,0801	1,0581	0,9096
0,75	1,1402	1,0795	0,8921
1,00	1,2247	1,0887	0,8889

Here, however, our task loses its sensible meaning, since negative lifts occur at the wing tips and consequently also negative bending moments, and of course the negative bending moments do not correspond to negative spar weights, but positive spar weights again. In the case of the sign change of M one would not have to take the integral over M but the integral over the absolute value of M and thus the basis of our whole calculation for this case becomes invalid. So the value of μ, which is reasonable in size, also represents the best value. In addition, according to the table of Yahls, the values of μ, which are considerably below 1, are not much worse. The elliptical lift distribution is, however, noticeably worse in the context of our task. The figure shows the circulation distributions according to Eq. 5 and the associated downward velocities according to Eq. 6 for a given value of the radius of gyration r and a given Γ_0. The curves **a** represent the elliptical lift distribution (μ = 0), the curves **b** and **c** belong to μ = 1/2 and μ = 1. The illustration shows that the recently favored Spityend wings. ??

deserve our point of view in preference to those with an approximately rectangular outline, but that in particular the degree of tapering does not matter too much.

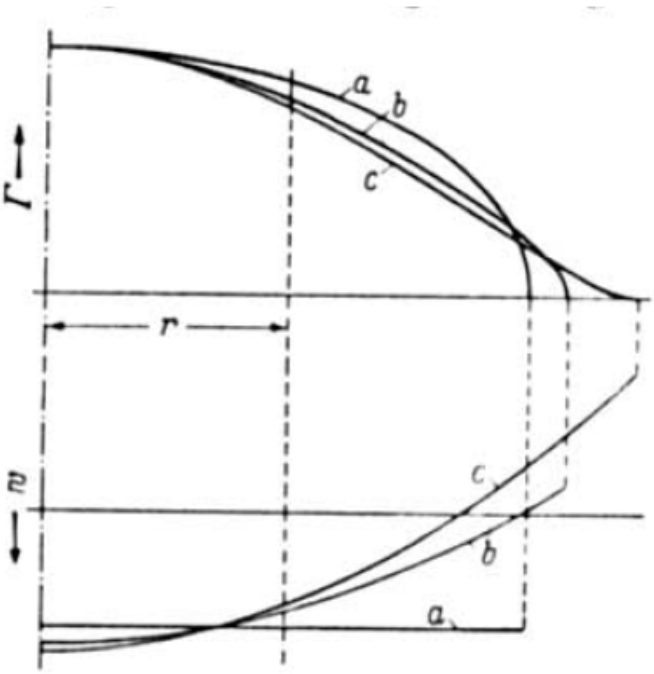

Abb. 1. Verlauf der Zirkulation und der Abwärtsgeschwindigkeit

3 – Summary

The object is to find the lift distribution which gives the smallest induced drag for a given total lift and a given moment of inertia of the total lift. This lift distribution is not elliptical, but rather corresponds to that of the pointed wing.